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THERMAL RESPONSE BEHAVIOR OF BOUNDARY-LAYER FLOWS

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Abstract—This paper presents an approximate, but general, analysis for the thermal response behavior of incompressible, constant property, laminar boundary layer flow over a smooth object of arbitrary shape. It encompasses the classic time-dependent Leveque problem as its special case. Comparisons with available data show that, for fluids with Prandtl number of the order of unity or larger, this analytical solution is able to provide reasonably accurate results for most engineering applications. Under certain restrictive conditions, it can also be used to predict the thermal response of a compressible boundary layer flow.

NOMENCLATURE

G_n, functions defined in (26), $n = 0, 1, 2, \ldots$;
g_i, functions defined in (27), $i = 1, 2, 3, 4$;
k, thermal conductivity;
Pr, Prandtl number;
T, temperature;
t, time;
U, velocity at edge of boundary layer;
u, velocity component in x-direction;
u_i, functions defined in (22), $i = 0, 1, 2, \ldots$;
v, velocity component in y-direction;
X, transformed coordinate defined in (12a) or (47a);
x, coordinate along body surface and in streamwise direction;
x_0, location where surface temperature undergoes a step change;
y, coordinate normal to x;
$\operatorname{erfc} x$, complementary error function, $= \dfrac{2}{\pi^{1/2}} \displaystyle\int_x^\infty e^{-\alpha^2} d\alpha$;
$i^n \operatorname{erfc} x$, nth repeated integral of the complementary error function, $= \displaystyle\int_x^\infty i^{n-1} \operatorname{erfc} \alpha \, d\alpha \quad n = 1, 2, \ldots$.

Greek symbols

α, thermal diffusivity;
β, wedge angle divided by π;
$\Gamma(n)$, gamma function, $= \displaystyle\int_0^\infty \alpha^{n-1} e^{-\alpha} d\alpha$;
$\Gamma_x(n)$, incomplete gamma function, $= \displaystyle\int_0^x \alpha^{n-1} e^{-\alpha} d\alpha$;
η, transformed coordinate defined in (12b) or (47b);
η_1, the variable η defined in [3] or [9];
θ, dimensionless temperature defined in (7);
$\bar{\theta}$, Laplace transform of θ defined in (17);
λ, a parametric function defined in (20);
λ_0, λ evaluated at wall;
μ, dynamic viscosity;
ν, kinematic viscosity;
τ, dimensionless time defined in (12c) or (47c);
ρ, density;
τ_w, wall shear;
τ_1, the variable τ defined in [3] or [9];
ψ, stream function;
ω, function defined in (13) or (48).

Superscript

$'$, differentiation with respect to x or η.

Subscripts

s, steady state condition;
w, condition at wall;
∞, free stream condition.

1. INTRODUCTION

A TECHNIQUE for solving unsteady, laminar, energy boundary-layer equation was introduced by Chao and Cheema [1] in 1968. It was first applied to boundary-layer flows over a flat plate and the unsteadiness was caused by a thermal disturbance at the plate surface. It has since been used in analyzing the transient heat- and mass-transfer behavior of translating droplets [2] and in the prediction of thermal response behavior of the Falkner–Skan flow [3]. The solution technique introduces a function, initially unknown, into the series expansion with the consequence that the solution becomes valid for all times. To evaluate this initially unknown function, one must separately determine the steady state solution.

In this paper we shall analyze the transient response time of laminar boundary-layer flows due to a sudden change in the temperature of its bounding surface. The analysis is approximate, yet general, and may be served

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as a basic solution for problems involving nonuniform surface temperature distribution. The method of analysis parallels that used in [1].

2. ANALYSIS

We consider a smooth object of arbitrary shape situated in an isothermal incoming stream of undisturbed temperature T_∞. A front portion of the surface, extending from the forward stagnation point (or the leading edge of a flat plate) to an arbitrary distance x_0, remains at constant wall temperature T_∞. At time $t = 0$ and for $x > x_0$, the wall temperature undergoes a step change from T_∞ to a constant value T_w. The flow is assumed to be steady, laminar, and incompressible and dissipation is to be ignored. The governing equation for the unsteady temperature boundary layer is

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2},\quad x \geqslant x_0,\ y > 0,\ t > 0. \tag{1}$$

The associated initial, boundary, and entrance conditions are

$$T(x, y, 0) = T_\infty \tag{2}$$

$$T(x, 0, t) = T_w,\quad T(x, \infty, t) = T_\infty \tag{3a, b}$$

$$T(x_0, y, t) = T_\infty. \tag{4}$$

Earlier investigations [3–6] have established that the initial growth of the thermal boundary layer is controlled by a molecular diffusion process. Its final decay to the steady state is governed by conditions near the wall. The latter was first pointed out by Riley [5]. Thus, in order to keep the analysis reasonably simple, it seems appropriate that we consider a one term representation of the velocity-distribution,* namely,

$$u(x, y) = \frac{\tau_w}{\mu}y. \tag{5}$$

It follows from the requirement of continuity that,

$$v(x, y) = -\frac{1}{2\mu}\frac{d\tau_w}{dx}y^2. \tag{6}$$

Introducing the dimensionless temperature function

$$\theta(x, y, t) = \frac{T - T_\infty}{T_w - T_\infty} \tag{7}$$

and making use of (5) and (6), (1) can be rewritten as

$$\frac{\partial\theta}{\partial t} + \frac{\tau_w}{\mu}y\frac{\partial\theta}{\partial x} - \frac{\tau_w'}{2\mu}y^2\frac{\partial\theta}{\partial y} = \alpha\frac{\partial^2\theta}{\partial y^2} \tag{8}$$

where the prime denotes differentiation with respect to x. The associated initial, boundary, and entrance conditions are

$$\theta(x, y, 0) = 0 \tag{9}$$

$$\theta(x, 0, t) = 1,\quad \theta(x, \infty, t) = 0 \tag{10a, b}$$

$$\theta(x_0, y, t) = 0. \tag{11}$$

To facilitate analysis, we consider the following transformation

$$\left.\begin{matrix} x \\ \\ y \\ \\ t \end{matrix}\right\} \rightarrow \left\{\begin{aligned} & X(x) = \ln\int_{x_0}^{x}\left(\frac{\tau_w}{\mu}\right)^{1/2}dx && (12a) \\ & \eta = y\left(\frac{\tau_w}{\mu}\right)^{1/2}\left[9\alpha\int_{x_0}^{x}\left(\frac{\tau_w}{\mu}\right)^{1/2}dx\right]^{-1/3} && (12b) \\ & \tau = \frac{\alpha\tau_w}{\mu}t\left[9\alpha\int_{x_0}^{x}\left(\frac{\tau_w}{\mu}\right)^{1/2}dx\right]^{-2/3} && (12c) \end{aligned}\right.$$

and denote

$$\omega(x) = \frac{\tau_w'\int_{x_0}^{x}\tau_w^{1/2}dx}{2\tau_w^{3/2}} = \frac{1}{2}\frac{\tau_w'}{\tau_w}\left(\frac{dX}{dx}\right)^{-1}. \tag{13}$$

It can then be established that (8) becomes

$$\frac{\partial^2\theta}{\partial\eta^2} + 3\eta^2\frac{\partial\theta}{\partial\eta} - 9\eta\frac{\partial\theta}{\partial X} = [1 + 6(3\omega - 1)\eta\tau]\frac{\partial\theta}{\partial\tau} \tag{14}$$

with

$$\theta(X, \eta, 0) = 0 \tag{15}$$

and

$$\theta(X, 0, \tau) = 1,\quad \theta(X, \infty, \tau) = 0. \tag{16a, b}$$

It is recognized that η is the similarity variable of the Lighthill problem [7]. The choice of $X(x)$ is not unique. It may differ by an arbitrary constant without affecting the remaining analysis. With η so selected, the entrance condition (11) merges with boundary condition (10b). The variable τ is recognized as the Fourier number of the problem.

Equation (14) subject to initial and boundary conditions (15) and (16a, b) may be solved by the series solution method first introduced by Chao and Cheema [1]. We define the Laplace transform of $\theta(X, \eta, t)$, in the usual way, i.e.

$$\bar\theta(X, \eta; P) = \int_0^\infty e^{-P\tau}\theta(X, \eta, \tau)d\tau \tag{17}$$

and obtain, from (14), (15), and (16a, b)

$$\frac{\partial^2\bar\theta}{\partial\eta^2} + 3\eta^2\frac{\partial\bar\theta}{\partial\eta} - 9\eta\frac{\partial\bar\theta}{\partial X} = P\bar\theta - 6(3\omega - 1)\eta\frac{\partial}{\partial P}(P\bar\theta) \tag{18}$$

with

$$\bar\theta(x, 0) = P^{-1},\quad \bar\theta(X, \infty) = 0. \tag{19a, b}$$

By following the procedure expounded in [1], we establish that an appropriate series solution for (18) and (19a, b) is

$$\bar\theta = P^{-1}\exp[-\tfrac{3}{2}\omega\eta^3 - (P + \lambda)^{1/2}\eta] \times \sum_{n=0}^{\infty}u_n(X, \eta)(P + \lambda)^{-n/2} \tag{20}$$

in which $Re(P) > 0$ and λ is real function of X and η, always positive, yet unknown. We set $u_0 \equiv 1$ and $u_1(X, 0) = u_2(X, 0) = \ldots = u_n(X, 0) = 0$, hence $\bar\theta(X, 0) = P^{-1}$. We shall later demonstrate that λ is such that $\bar\theta(X, \infty) = 0$.

*Analysis based on two-term representation of the velocity-distribution, namely

$$u = \frac{\tau_w}{\mu}y + \frac{1}{2\mu}\frac{dp}{dx}y^2,$$

has also been made. It is felt that the extreme complication does not justify its presentation at the present time.

Upon substituting (20) into (18) and equating the coefficients of like power of $(P+\lambda)$, we find,

$$
\begin{aligned}
2\frac{\partial u_n}{\partial \eta} = {}& \frac{\partial^2 u_{n-1}}{\partial \eta^2} - (9\omega-3)\eta^2\frac{\partial u_{n-1}}{\partial \eta} - 9\eta\frac{\partial u_{n-1}}{\partial X} - \frac{\partial \lambda}{\partial \eta}\left[\eta\frac{\partial u_{n-2}}{\partial \eta} + (n-3)\frac{\partial u_{n-3}}{\partial \eta}\right] \\
& - \left[9\omega\eta - \frac{27}{2}\omega\left(\frac{3}{2}\omega-1\right)\eta^4 - \eta\frac{\partial \lambda}{\partial \eta} - \frac{27}{2}\frac{\mathrm{d}\omega}{\mathrm{d}X}\eta^4 - \lambda + \frac{n-1}{2}(18\omega-6)\eta\right]\cdot u_{n-1} \\
& - \left[\frac{1}{2}\eta\frac{\partial^2\lambda}{\partial \eta^2} - 3\left(\frac{3}{2}\omega - \frac{1}{2}\right)\eta^3\frac{\partial \lambda}{\partial \eta} - (n-3)\frac{\partial \lambda}{\partial \eta} - \frac{1}{2}(18\omega-6)\cdot\eta^2\lambda - \frac{9}{2}\eta^2\frac{\partial \lambda}{\partial X}\right]\cdot u_{n-2} \\
& - \left\{\frac{n-3}{2}\cdot\left[\frac{\partial^2\lambda}{\partial \eta^2} - 3(3\omega-1)\eta^2\frac{\partial \lambda}{\partial \eta} - 9\eta\frac{\partial \lambda}{\partial X} - (18\omega-6)\eta\lambda\right] - \frac{1}{4}\eta^2\left(\frac{\partial \lambda}{\partial \eta}\right)^2\right\}\cdot u_{n-3} \\
& + \frac{2n-7}{4}\eta\left(\frac{\partial \lambda}{\partial \eta}\right)^2\cdot u_{n-4} + \frac{1}{4}(n-3)(n-5)\left(\frac{\partial \lambda}{\partial \eta}\right)^2 u_{n-5}
\end{aligned}
\tag{21}
$$

for $n \geqslant 1$ and $u_{-1} = u_{-2} = u_{-3} = u_{-4} = 0$. The recurrent relation (21) can be integrated in succession to obtain the following expressions for $u_n(X, \eta)$

$$u_0 = 1 \tag{22a}$$

$$u_1 = H_1 + \tfrac{1}{2}\eta\lambda \tag{22b}$$

$$u_2 = H_2 + \tfrac{1}{2}H_1\eta\lambda + \tfrac{1}{8}(\eta\lambda)^2 \tag{22c*}$$

where

$$H_1 = -\frac{9}{4}\omega\eta^2 + \frac{27}{20}\eta^5\left(\frac{3}{2}\omega^2 - \omega + \frac{\mathrm{d}\omega}{\mathrm{d}X}\right) \tag{23a}$$

$$
\begin{aligned}
H_2 = {}& \frac{1}{2}\frac{\partial H_1}{\partial \eta} + \frac{1}{2}H_1^2 - \frac{3}{2}(3\omega-1)H_1\eta^2 \\
& + \frac{27}{32}\left(3\frac{\mathrm{d}\omega}{\mathrm{d}X} - 3\omega^2 + \omega\right)\eta^4 + \frac{81}{280}\left(\frac{9}{2}\omega^3 - \frac{9}{2}\omega^2\right. \\
& \left. + \omega + 2\frac{\mathrm{d}\omega}{\mathrm{d}X} - 6\omega\frac{\mathrm{d}\omega}{\mathrm{d}X} - 3\frac{\mathrm{d}^2\omega}{\mathrm{d}X^2}\right)\eta^7.
\end{aligned}
\tag{23b}
$$

Accordingly, we have,

$$
\left.
\begin{aligned}
& \frac{\partial u_0}{\partial \eta} \equiv 0,\ \frac{\partial u_1}{\partial \eta}(X,0) = \frac{1}{2}\lambda_0,\ \frac{\partial u_2}{\partial \eta}(X,0) = -\frac{9}{4}\omega \\
& \frac{\partial u_3}{\partial \eta}(X,0) = \frac{1}{8}\lambda_0^2,\ \frac{\partial u_4}{\partial \eta}(X,0) = -\frac{9}{4}\lambda_0\omega, \ldots
\end{aligned}
\right\}
\tag{24}
$$

It is observed that, with the exception of u_0 (which is identically unity) each u_n consists of two parts—one is free of λ and is designated as H_n in (22); the other is a polynomial of λ.

The desired transient temperature field is obtained from taking the inverse of (20), using the familiar translation and convolution theorems. The result is:

$$\theta(X, \eta, \tau) = \exp[-\tfrac{3}{4}\omega\eta^3 - \eta\sqrt{\lambda}]\cdot\sum_{n=0}^{\infty} u_n\lambda^{-n/2}G_n \tag{25}$$

*The author has also evaluated u_3 and u_4. They are omitted from the list for the interest of conserving space.

in which,

$$
\left.
\begin{aligned}
& G_0 = g_1 + g_2,\ G_1 = g_1 - g_2,\ G_2 = G_0 - g_3 \\
& G_3 = G_1 + \eta\lambda^{1/2}g_3 - g_4, \\
& G_4 = G_0 - (1 + \tau\lambda + \tfrac{1}{2}\eta^2\lambda)g_3 + \tfrac{1}{2}\eta(\lambda)^{1/2}g_4 \\
& G_5 = G_1 + (1 + \tau\lambda + \tfrac{1}{6}\eta^2\lambda)\eta(\lambda)^{1/2}g_3 \\
& \quad - (1 + \tfrac{2}{3}\lambda\tau + \tfrac{1}{6}\eta^2\lambda)g_4, \ldots \text{etc.}
\end{aligned}
\right\}
\tag{26}
$$

with,

$$
\left.
\begin{aligned}
& g_1 = \tfrac{1}{2}\mathrm{erfc}\left(\frac{\eta}{2(\tau)^{1/2}} - (\lambda\tau)^{1/2}\right), \\
& g_2 = \tfrac{1}{2}\mathrm{e}^{2\eta\sqrt{(\lambda)}}\mathrm{erfc}\left(\frac{\eta}{2\sqrt{(\tau)}} + (\lambda\tau)^{1/2}\right) \\
& g_3 = \mathrm{e}^{\eta(\lambda)^{1/2} - \lambda\tau}\mathrm{erfc}\left(\frac{\eta}{2(\tau)^{1/2}}\right), \\
& g_4 = \frac{2}{\pi^{1/2}}\cdot(\tau\lambda)^{1/2}\,\mathrm{e}^{-\left[\frac{\eta}{2(\tau)^{1/2}} - (\lambda\tau)^{1/2}\right]^2}.
\end{aligned}
\right\}
\tag{27}
$$

For $n \geqslant 2$, G_n is given by the integral:

$$G_n = 2^{n-2}e^{\eta(\lambda)^{1/2}}\int_0^{\lambda\tau}\beta^{n/2-1}\mathrm{e}^{-\beta}i^{n-2}\mathrm{erfc}\times\left(\frac{\eta}{2}\lambda^{1/2}\beta^{-1/2}\right)\mathrm{d}\beta. \tag{28}$$

As it turns out that the G_n-functions are all expressible in terms of the four g-functions. Furthermore, it can be demonstrated that:

(i) $\lim_{\tau\to\infty} G_n = 1$ for all η's

(ii) $\lim_{\tau\to 0} G_n = 0$ for all η's

(iii) For any fixed τ, as $\eta \to 0$

$$G_0 = 1,\quad G_1 = \mathrm{erf}(\lambda_0\tau)^{1/2},$$

$$G_2 = 1 - \mathrm{e}^{-\lambda_0\tau}$$

$$G_3 = \mathrm{erf}(\lambda_0\tau)^{1/2} - \frac{2}{\pi^{1/2}}(\lambda_0\tau)^{1/2}\mathrm{e}^{-\lambda_0\tau},$$

$$G_4 = 1 - (1 + \tau\lambda_0)\mathrm{e}^{-\lambda_0\tau}$$

$$G_5 = \mathrm{erf}(\lambda_0\tau)^{1/2} - (1 + \tfrac{2}{3}\lambda_0\tau)\times\frac{2}{\pi^{1/2}}(\tau\lambda_0)^{1/2}\mathrm{e}^{-\lambda_0\tau}, \ldots$$

which are all positive and less than unity except for G_0

(iv) $\lim_{\eta\to\infty} G_n = 0$ for all τ's.

Thus, one may conclude that G_n ranges from 0 to 1 for the entire domain of interest, namely, $0 < \tau < \infty$ and $0 < \eta < \infty$.

The steady state temperature distribution follows from letting $\tau \to \infty$ in (25). It is

$$\theta_s(X, \eta) = \exp[-\tfrac{1}{3}\omega\eta^3 - \eta(\lambda)^{1/2}] \cdot \sum_{n=0}^{\infty} u_n \lambda^{-n/2}. \tag{29}$$

Since $\theta_s(X, \infty) = 0$ and $Re(P) > 0$, we conclude, by comparing (29) with (20), that $\bar{\theta}(X, \infty) = 0$. Thus, an earlier expectation is realized. From the analysis of Acrivos [8], it is known that θ_s depends on η only. This fact can also be deduced from (14) subject to conditions (15) and (16a, b). Consequently, in (29) θ_s is actually independent of X. The unknown function $\lambda(X, \eta)$ must be determined in such a way that the RHS of (29) is independent of X. A rigorous mathematical proof of the existence of such λ function is unknown. However, computer solutions for some physical meaningful cases show that they do indeed exist.

Differentiating (25) with respect to η and evaluating the result for $\eta = 0$ yield:

$$-\frac{\partial\theta}{\partial\eta}(X, 0, \tau) = \frac{1}{(\pi\tau)^{1/2}} e^{-\lambda_0\tau} + (\lambda_0)^{1/2} \operatorname{erf}(\lambda_0\tau)^{1/2}$$
$$\sum_{n=1}^{\infty} \lambda_0^{-n/2} \frac{\Gamma_{\lambda_0\tau}(n/2)}{\Gamma(n/2)} \cdot \frac{\partial u_n}{\partial\eta}(X, 0) \tag{30}$$

in which $\partial u_n/\partial\eta\,(X, 0)$ is given by (24). The steady state temperature derivative at the solid surface is

$$-\frac{d\theta_s}{d\eta}(0) = (\lambda_0)^{1/2} - \sum_{n=1}^{\infty} \lambda_0^{-n/2} \frac{\partial u_n}{\partial\eta}(X, 0). \tag{31}$$

The local transient surface heat flux is

$$q_w = -k(T_w - T_\infty)\left(\frac{\tau_w}{\mu}\right)^{1/2} \times \left[9\alpha \int_{x_0}^{x} \left(\frac{\tau_w}{\mu}\right)^{1/2} dx\right]^{-1/3} \frac{\partial\theta}{\partial\eta}(X, 0, \tau). \tag{32}$$

To determine the function $\lambda(X, \eta)$, we must separately evaluate the steady state temperature field θ_s. This has been already given by Lighthill [7]. Lighthill obtained the solution by solving the boundary-layer equations in the von Mises form using operational methods. An elegant way to obtain the same result using similarity consideration was given by Acrivos [8]. The steady state temperature field is

$$\theta_s(\eta) = 1 - \frac{3}{\Gamma(1/3)} \int_0^{\eta} e^{-\beta^3} d\beta. \tag{33}$$

Thus, it follows that

$$\theta'_s(\eta) = -\frac{3}{\Gamma(1/3)} e^{-\eta^3}. \tag{34}$$

The local heat flux is

$$q_{w,s} = \frac{3}{\Gamma(1/3)} k(T_w - T_\infty)\left(\frac{\tau_w}{\mu}\right)^{1/2} \times \left[9\alpha \int_{x_0}^{x} \left(\frac{\tau_w}{\mu}\right)^{1/2} dx\right]^{-1/3}. \tag{35}$$

By equating (29) and (33), one has the following expression for λ

$$\exp[-\tfrac{1}{3}\omega\eta^3 - \eta(\lambda)^{1/2}] \sum_{n=0}^{\infty} u_n \lambda^{-n/2} = 1 - \frac{3}{\Gamma(1/3)} \int_0^{\eta} e^{-\beta^3} d\beta. \tag{36}$$

At the wall, $\eta = 0$, the LHS of (36) becomes identically unity and hence it cannot be used to determine λ_0. The latter was evaluated from

$$(\lambda_0)^{1/2} - \sum_{n=1}^{\infty} \lambda_0^{-n/2} \frac{\partial u_n}{\partial\eta}(X, 0) = \frac{3}{\Gamma(1/3)}. \tag{37}$$

Note that the RHS of (36) can also be calculated by a direct numerical integration of

$$\theta''_s + 3\eta^2\theta'_s = 0 \text{ with } \theta_s(0) = 1 \text{ and } \theta'_s(0) = \frac{-3}{\Gamma(1/3)}$$

where primes denote differentiation with respect to η. For engineering applications, one is often concerned with the ratio of the instantaneous wall flux to its steady state value. This ratio is given by

$$\frac{q_w}{q_{w,s}} = \frac{\Gamma(1/3)}{3}\left[-\frac{\partial\theta}{\partial\eta}(X, 0, \tau)\right]. \tag{38}$$

To develop some feeling on the magnitude of the function ω, we consider wedge flow for which $U(x) = CX^m$, $m = \beta/(2-\beta)$. For attached boundary layer $m > -0.091$. A straight forward calculation yields

$$\omega(x) = 1 - \frac{4}{3(m+1)}\left\{1 + \frac{3m-1}{4}\left(\frac{x_0}{x}\right)^{(3/4)(m+1)}\right\}. \tag{39}$$

Thus, ω depends on x_0/x only. It does not depend on x or x_0 explicitly. For isothermal wedges, $x_0 \equiv 0$; and

$$\omega = \frac{3m-1}{3m+3} = \frac{2\beta-1}{3}, \text{ a constant.} \tag{40}$$

Equation (40) shows that $\omega = -1/3$ for a flat plate; $\omega = 0$ for flow over 90° wedge; and $\omega = 1/3$ for the two-dimensional stagnation flow (Hiemenz flow).

3. APPLICATIONS

Equation (37) has been solved for several selected values of ω and the results are displayed in Fig. 1. It is interesting to note that there is no real positive root when $\omega > 1/3$. However, it has double roots when $0 \leqslant \omega < 1/3$. For this double-root region, only the higher value of λ_0 should be used.

The values of λ_0 shown in Fig. 1 are inserted into (30) to calculate the wall derivative. The wall heat flux ratio is then evaluated from (38). The results are graphically shown in Fig. 2 for $\omega = -0.3$, 0, and 0.3.

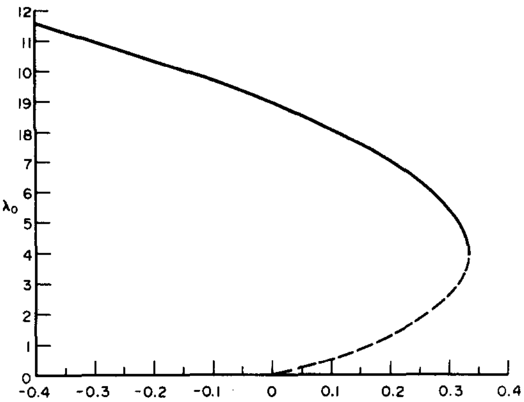

FIG. 1. Values of λ_0 as a function of ω.

steady state. When $\tau \geqslant 0.4$, the steady state condition prevails. Thus, for engineering application, the transient response time could be taken as $\tau = 0.4$.

When τ_w is a constant, $\omega = 0$; this is the time-dependent Leveque problem studied by Soliman and Chambre [6]. They obtained their solution by double Laplace transformation. Their expression for the ratio of the instantaneous wall flux to its steady state value is expressed by an infinite series. The first term of the series corresponds to the steady state part of the solution. Figure 3 compares their results with those obtained from present analysis, using two, three, and four non-zero terms in the present series. Note that, when $\omega = 0$, $u_2'(0)$, $u_4'(0)$, and $u_6'(0)$ all vanish identically. Consequently, $u_5'(0)$ and $u_7'(0)$ have also been

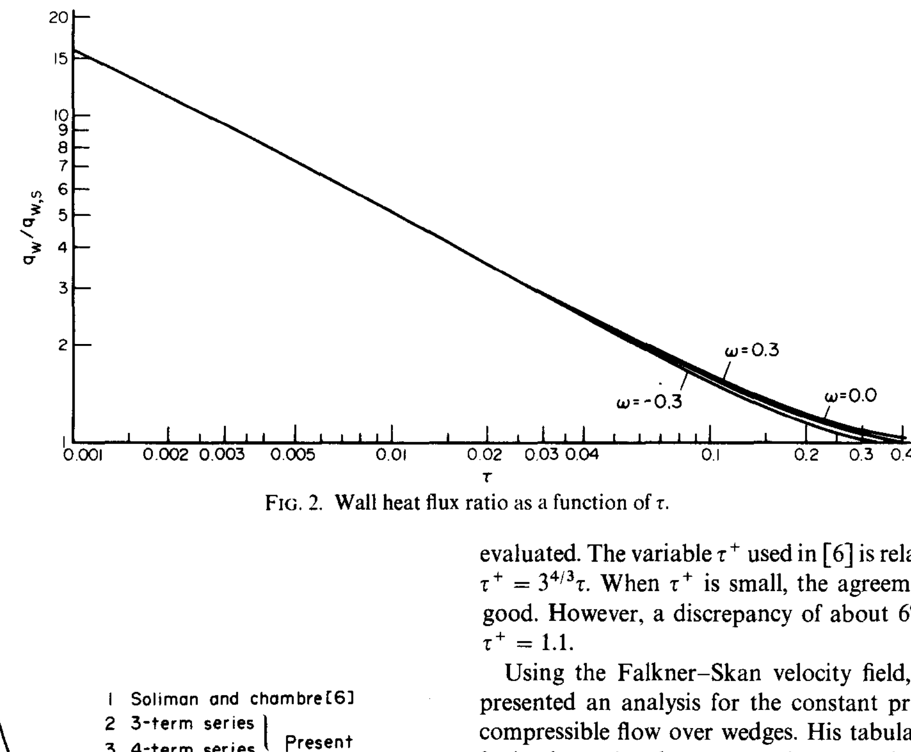

FIG. 2. Wall heat flux ratio as a function of τ.

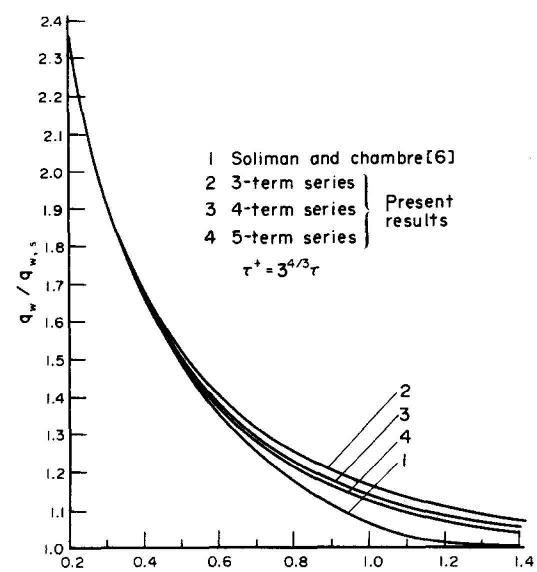

FIG. 3. Wall heat flux ratio for time-dependent Leveque problem.

As expected, the wall flux ratio is virtually independent of ω when τ is small. Fluid acceleration or deceleration and, hence, positive or negative values of ω exhibit their influence only in the final stage of decay to the

evaluated. The variable τ^+ used in [6] is related to τ by $\tau^+ = 3^{4/3}\tau$. When τ^+ is small, the agreement is very good. However, a discrepancy of about 6% exists at $\tau^+ = 1.1$.

Using the Falkner–Skan velocity field, Chen [9] presented an analysis for the constant property, incompressible flow over wedges. His tabulated results for isothermal wedges were replotted in Figs. 4a and b, and compared with present solutions for $Pr = 1$ and 100 and $\beta = -0.1$, and 0.5. Figure 5 compares present temperature fields with Chen's [9] for $Pr = 1$, 100 and $\beta = 0.5$. The agreement is probably as good as one may hope for. Note that in Figs. 4a–5, τ_1 and η_1 are respectively the variables τ and η in [3] and [9].

4. GENERALIZATION TO COMPRESSIBLE FLOWS

As was demonstrated in [1] that, under certain restrictive conditions, the incompressible results obtained in the previous sections can be directly used to describe the thermal response of compressible boundary-layer flows.

For compressible flow the governing boundary-layer equations are

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0 \tag{41}$$

$$C_p\rho\left(\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial \theta}{\partial y}\right). \tag{42}$$

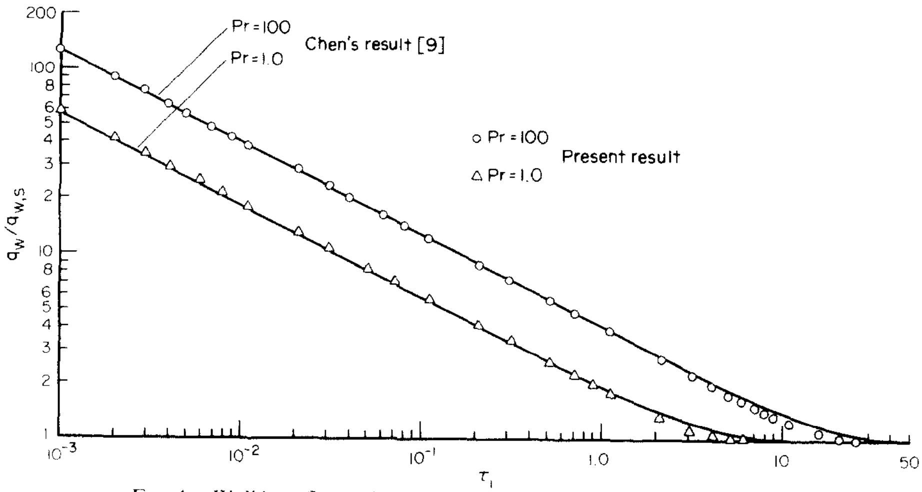

FIG. 4a. Wall heat flux ratio for isothermal wedge surface with $\beta = -0.1$.

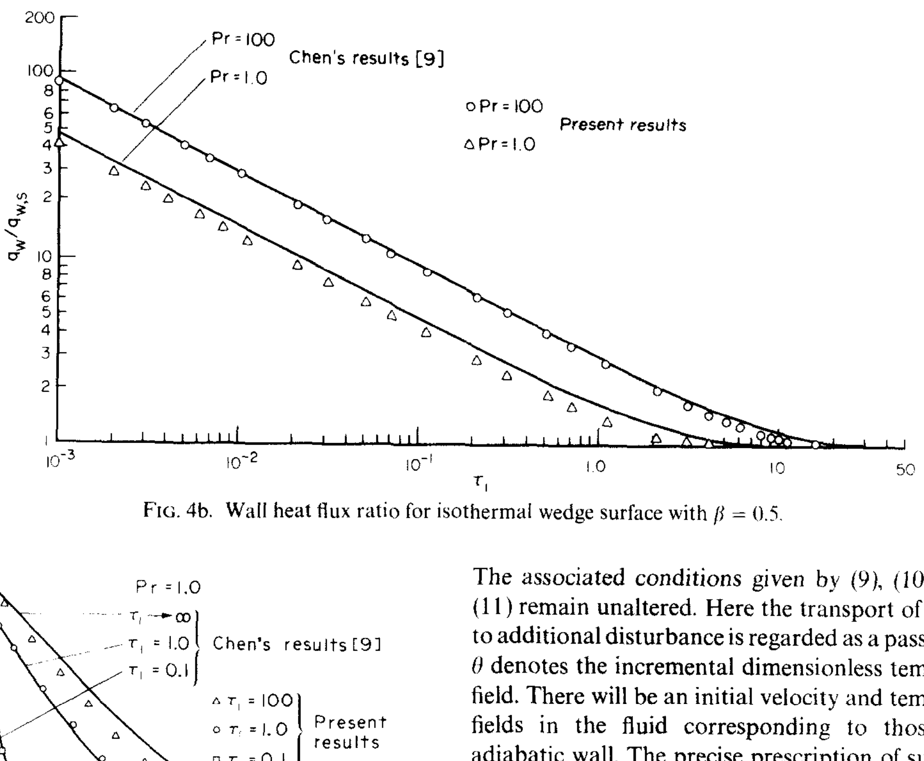

FIG. 4b. Wall heat flux ratio for isothermal wedge surface with $\beta = 0.5$.

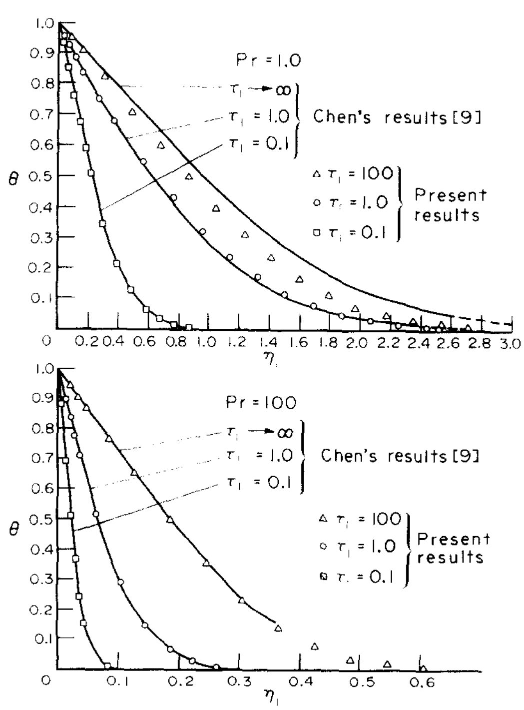

FIG. 5. Temperature profiles for isothermal wedge flows with $\beta = 0.5$.

The associated conditions given by (9), (10a, b) and (11) remain unaltered. Here the transport of heat due to additional disturbance is regarded as a passive scale. θ denotes the incremental dimensionless temperature field. There will be an initial velocity and temperature fields in the fluid corresponding to those of an adiabatic wall. The precise prescription of such fields would, in general, depend on the equation of state of the gas and the manner by which its viscosity, conductivity and specific heat vary with temperature.

In what follows we shall consider only an ideal gas of constant Pr and C_p with $\mu\rho = \mu_w\rho_w$ (a function of x only) and $k\rho = k_w\rho_w$ (also a function of x only). These assumptions are found to be reasonably accurate for common gases within moderate temperature ranges.

To effect solution, we now introduce a stream function $\psi(x, y, t)$ defined by

$$\rho u = \frac{\partial \psi}{\partial y}, \text{ and } -\rho v = \frac{\partial \psi}{\partial x} + \frac{\partial}{\partial t}\int_0^y \rho \, \mathrm{d}y. \quad (43\text{a, b})$$

and transform the (x, y, t) system to (x, z, t) system according to

$$z = \int_0^y \rho \, \mathrm{d}y. \quad (44)$$

Then, with the one term velocity-distribution

$$u = \frac{\tau_w}{\rho_w \mu_w} z \tag{45}$$

it can be shown that (42) becomes

$$\frac{\partial \theta}{\partial t} + \frac{\tau_w}{\rho_w \mu_w} z \frac{\partial \theta}{\partial x} - \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\tau_w}{\rho_w \mu_w}\right)\right] z^2 \frac{\partial \theta}{\partial z} = \frac{\rho_w k_w}{C_p} \frac{\partial^2 \theta}{\partial z^2}. \tag{46}$$

With the following transformation,

$$\left.\begin{matrix} x \\ \\ z \\ \\ t \end{matrix}\right\} \left\{\begin{aligned} X &= \ln \int_{x_0}^{x} \frac{k_w \rho_w}{C_p}\left(\frac{\tau_w}{\rho_w \mu_w}\right)^{1/2} \mathrm{d}x && (47\mathrm{a}) \\ \eta &= z\left(\frac{\tau_w}{\rho_w \mu_w}\right)^{1/2}\left[9 \int_{x_0}^{x} \frac{\rho_w \mu_w}{C_p}\left(\frac{\tau_w}{\rho_w \mu_w}\right)^{1/2} \mathrm{d}x\right]^{-1/3} && (47\mathrm{b}) \\ \tau &= \frac{\rho_w k_w}{C_p} \cdot t \cdot \left(\frac{\tau_w}{\rho_w \mu_w}\right)\left[9 \int_{x_0}^{x} \frac{\rho_w k_w}{C_p}\left(\frac{\tau_w}{\rho_w \mu_w}\right)^{1/2} \mathrm{d}x\right]^{-2/3} && (47\mathrm{c}) \end{aligned}\right.$$

equation (46) can be transformed to equation (14) if ω is redefined as

$$\omega = \frac{\dfrac{\mathrm{d}\tau_w}{\mathrm{d}x}}{2\tau_w \dfrac{\mathrm{d}X}{\mathrm{d}x}}. \tag{48}$$

Since the conditions given by (15), and (16a,b) remain unaltered, the incompressible results given in the previous sections are directly applicable to the present case. Note that the variable η defined by (47b) is the similarity variable for the corresponding steady state problem.

5. CONCLUDING REMARKS

The time-dependent Lighthill's problem has been analyzed by the technique developed by Chao and Cheema [1]. This general, but approximate, analysis can predict not only the transient surface heat flux, but also the transient temperature field. It is directly applicable to surfaces with either a uniform temperature ($x_0 = 0$) or a step discontinuity in temperature. Since the energy equation and its associated conditions are linear, it is also applicable to objects with any arbitrarily prescribed surface temperature distribution if the Duhamel's procedure is used. In addition, it is also applicable to non-Newtonian flows.

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REPONSE THERMIQUE D'ECOULEMENTS A COUCHE LIMITE

Résumé—On présente une analyse approchée mais générale de la réponse thermique d'une couche limite laminaire incompressible, à propriété constantes, sur un objet lisse de forme quelconque. Elle englobe le problème classique de Leveque, comme cas particulier. Des comparaisons avec les données disponibles montrent que, pour des fluides à nombre de Prandtl de l'ordre de l'unité ou plus grand, cette solution analytique est capable de fournir des résultats suffisamment précis pour des applications pratiques. Sous des conditions restrictives, elle peut aussi être utilisée pour estimer la réponse thermique d'une couche limite compressible.

DAS TEMPERATUR-ZEIT-VERHALTEN VON GRENZSCHICHTSTRÖMUNGEN

Zusammenfassung—Diese Arbeit stellt eine angenäherte, aber allgemeine Berechnung des Temperatur-Zeit-Verhaltens einer inkompressiblen, laminaren Grenzschichtströmung mit konstanten Stoffwerten über glatte Gegenstände von beliebiger Form vor. Als Spezialfall beinhaltet sie das klassische zeitabhängige Leveque-

Problem. Vergleiche mit verfügbaren Meßwerten zeigen, daß die analytische Lösung für Fluide mit einer Prandtl-Zahl in der Größenordnung von 1 oder größer genügend genaue Ergebnisse für die meisten ingenieurtechnischen Anwendungen liefern kann. Mit gewissen Einschränkungen kann sie auch verwendet werden, um das Temperatur-Zeit-Verhalten der kompressiblen Grenzschichtströmung abzuschätzen.

ВЛИЯНИЕ ТЕПЛОВОГО ВОЗДЕЙСТВИЯ НА ТЕЧЕНИЕ В ПОГРАНИЧНОМ СЛОЕ

Аннотация — Дан приближенный достаточно общий анализ теплового воздействия на ламинарное течение несжимаемой жидкости с постоянными свойствами в пограничном слое на гладкой поверхности тела произвольной формы. Как частный случай анализ включает классическую нестационарную задачу Левека. Сравнение с имеющимися данными показывает, что для жидкостей с числом Прандтля порядка единицы или больше это аналитическое решение может дать достаточно точные результаты для многих технических приложений. При определенных ограничениях его можно использовать также для расчета теплового воздействия на пограничный слой сжимаемой жидкости.